ACADEMIC PRESS
Journal of Number Theory 97 (2002) 447–471
JOURNAL OF Number Theory
www.academicpress.com

A Galois criterion for good reduction of τ-sheaves

Francis Gardeyn[a,b,*,1]

[a] *ETH Zürich, D-MATH, Rämistrasse 101, 8092 Zürich, Switzerland*
[b] *Universiteit Gent, Vakgroep ZWC, Galglaan 2, 9000 Gent, Belgium*

Received 18 January 2002; revised 11 April 2002
Communicated by David Goss

Abstract

Let R be a complete discrete valuation $\mathbb{F}_q$-algebra with fraction field K and perfect residue field k. For an irreducible smooth affine curve C, with field of constants $\mathbb{F}_q$, let M denote a τ-sheaf over C_K, endowed with a characteristic morphism ι: Spec $K \to C$. Given a Tate module $T_\ell(M)$ with trivial action of the inertia group I_K, we construct a good model $\mathcal{M}$ for M over C_R. This yields an analog for τ-sheaves of the classical Néron–Ogg–Shafarevič theorem on good reduction of abelian varieties.

We can actually extend this result to a criterion for nondegenerate and semistable reduction. As an application, we show how the local L-factor of a τ-sheaf at a place of bad reduction is related to the action of Frobenius on the associated Galois representations. Finally, we discuss the implications of these results to Drinfeld modules and their associated t-motives.

MSC: 11G09

Keywords: τ-Sheaves; Good reduction; Galois representations

*Corresponding author. Aprikosenstrasse 22, Zürich 8051, Switzerland.
E-mail address: fgardeyn@math.ethz.ch, fgardeyn@freesurf.ch.
[1] The author is a Research Assistant of the Fund for Scientific Research Flanders. The present work is part of a project supported by the Swiss Science Foundation.

PII: S0022-314X(02)00014-8

1. Introduction

1.1. Models for τ-sheaves

For a finite field $\mathbb{F}_q$ with q elements, let R be a complete discrete valuation $\mathbb{F}_q$-algebra with a perfect residue field k. Let us denote the fraction field of R by K, a uniformizer by π and the valuation by v. Let x be the closed point of $\operatorname{Spec} R$ and Γ_K (resp. I_K) the absolute Galois group of K (resp. its inertia subgroup).

Consider an irreducible smooth affine curve C with field of constants $\mathbb{F}_q$, and let $\mathbf{A}$ be its ring of global regular functions. For X equal to K or R, let C_X denote the scheme $C \times \operatorname{Spec} X$. Further, we fix a morphism $\imath : \operatorname{Spec} K \to C$. If $\imath$ can be extended to a morphism $\operatorname{Spec} R \to C$, then we say that the valuation of K is finite, and we denote the closed point $\imath(x)$ of C by ℓ_x.

Denote by $\varphi \in \operatorname{End}(X)$ the Frobenius morphism defined by the homomorphism

$$X \to X : x \mapsto x^q.$$

We endow the scheme C_X with the endomorphism $\sigma := id \times \varphi$. Let $\mathcal{O}_{C_X}$ denote the structure sheaf of the scheme C_X.

We recall (cf. [Ga1] Section 1) that we define a *τ-sheaf* (M, τ) (for short: M) *of rank* $r \geqslant 1$, *defined on* C_K (*resp.* C_R), as a *locally free* $\mathcal{O}_{C_K}$-module (resp. $\mathcal{O}_{C_R}$-module) of finite rank r, endowed with an *injective* morphism

$$\tau : \sigma^* M \to M.$$

A τ-sheaf M on C_K is said to have *characteristic* $\imath$ if the cokernel of τ is supported on the point $\Gamma(\imath)$ of C_K which is defined as the graph of $\imath$.

A *model* $\mathcal{M}$ *on* C_R of a τ-sheaf M on C_K is a τ-sheaf on C_R which extends M, i.e. such that M is the generic fibre of $\mathcal{M}$. In [Ga1] Proposition 2.13, we proved that there exists a *maximal* model $\mathcal{M}^{\max}$ of M on C_R, unique up to unique isomorphism, which contains all other models. Consider the *reduction* $\bar{\mathcal{M}}$ of $\mathcal{M}$ on the special fibre C_k of C_R, a locally free coherent sheaf endowed with the induced homomorphism $\tau : \sigma^* \bar{\mathcal{M}} \to \bar{\mathcal{M}}$. The τ-sheaf $\mathcal{M}$ is called *good* if $\bar{\mathcal{M}}$ is a τ-sheaf (i.e. τ is injective) and *nondegenerate* if $\bar{\mathcal{M}}$ contains a nonzero τ-sheaf. The maximal rank of such a τ-sheaf is called the *nondegenerate rank* of $\mathcal{M}$ and $\bar{\mathcal{M}}$.

Let ℓ be a closed point of C. We denote by $\mathbf{A}_\ell$ the completion with respect to the ℓ-adic topology of the local ring of regular functions on C at ℓ. For every ℓ, one can associate to M in a covariant functorial way a free $\mathbf{A}_\ell$-module $H_\ell(M)$, endowed with a continuous action by the absolute Galois group Γ_K of K (see Section 2). This module is the dual of the *Tate module* $T_\ell(M)$:

$$H_\ell(M) := \operatorname{Hom}_{\mathbf{A}_\ell}(T_\ell(M), \mathbf{A}_\ell).$$

If M has rank r, then the $\mathbf{A}_\ell$-rank of $H_\ell(M)$ is r for all closed points ℓ of $C \backslash \{\ell_x\}$.

Related to Drinfeld's shtuka's, τ-sheaves (also refered to as φ-sheaves) are considered basic objects in the arithmetic of function fields, and this paper is written from that perspective. The reader that is more familiar with Drinfeld modules (and their generalizations, Anderson's abelian t-modules) might want to skip ahead to Section 8, where we recall how τ-sheaves are naturally related to them, and, at the same time, illustrate the usefulness of taking the τ-sheaf approach.

1.2. Galois criterion

In this paper, we establish an analog of the Néron–Ogg–Shafarevič criterion for good reduction of abelian varieties (cf. [BLR] 7.4, Theorem 5) for τ-sheaves on C_K:

Theorem 1.1 (Galois criterion for good reduction). *Let M be a τ-sheaf on C_K with characteristic ι. Then the following statements are equivalent*:

(i) *There exists a good model $\mathscr{M}$ for M on C_R.*
(ii) *For all closed points ℓ of $C\backslash\{\ell_x\}$, the Tate module $T_\ell(M)$ (or its dual $\mathbf{A}_\ell[\Gamma_K]$-module $H_\ell(M)$) is unramified.*
(iii) *There exists a closed point ℓ of $C\backslash\{\ell_x\}$ such that the Tate module $T_\ell(M)$ (or its dual $\mathbf{A}_\ell[\Gamma_K]$-module $H_\ell(M)$) is unramified.*

We will actually obtain the following stronger result:

Theorem 1.2 (Galois criterion for nondegenerate reduction). *Let M be a τ-sheaf on C_K of rank r with characteristic ι. Then the following statements are equivalent*:

(i) *There exists a model $\mathscr{M}$ for M on C_R with nondegenerate rank at least r'.*
(ii) *For all but a finite number of closed points ℓ of C, the $\mathbf{A}_\ell[\Gamma_K]$-module $H_\ell(M)$ contains an unramified submodule H which is free over $\mathbf{A}_\ell$ of rank at least r'.*
(iii) *There exists a closed point ℓ of C such that the $\mathbf{A}_\ell[\Gamma_K]$-module $H_\ell(M)$ has $\mathbf{A}_\ell$-rank r and contains an unramified submodule H which is free over $\mathbf{A}_\ell$ of rank at least r'.*

The proof of these theorems will take up Sections 2–5. As we remark in Section 6, there exists a version of Theorem 1.2 for analytic τ-sheaves which yields a Galois criterion for the *semistability* of the τ-sheaf.

In Section 7, we show how, as a consequence of Theorem 1.2, for a given τ-sheaf M, the local factor at x of the L-function of M, defined via the action of Frobenius on the associated Galois modules $H_\ell(M)$, can be proved to have coefficients in $\mathbf{A}$ and to be independent of the closed point ℓ, for almost all closed points ℓ of C.

In the last section, Section 8, we explain the connections between the reduction theories for Drinfeld modules and Anderson t-modules and their associated t-motives.

2. ℓ-Adic τ-sheaves and Galois representations

2.1. Reformulation of Theorem 1.2

First, we want to reformulate the above statements in terms of ℓ-adic τ-sheaves. Letting X denote either K or R, we let $\hat{C}_{X,\ell}$ be the formal completion of the scheme C_X along $\{\ell\} \times \operatorname{Spec} X$ and $\mathcal{O}_{\hat{C}_{X,\ell}}$ its structure ring. The Frobenius morphism φ on X induces the endomorphism $\sigma := id \times \varphi$ on $\hat{C}_{X,\ell}$. A τ-sheaf $\hat{M}_\ell$ on $\hat{C}_{X,\ell}$ (also called an *ℓ-adic τ-sheaf*) is called *smooth* if τ acts as an isomorphism on $\hat{M}_\ell$.

Let $\hat{M}_\ell$ be a smooth τ-sheaf on $\hat{C}_{K,\ell}$ and $\hat{\mathcal{M}}_\ell$ a model for $\hat{M}_\ell$ over $\hat{C}_{R,\ell}$. We say that $\hat{\mathcal{M}}_\ell$ has *nondegenerate rank* $\rho \geqslant 1$ if its reduction $\bar{\mathcal{M}}_\ell$ to $\hat{C}_{k,\ell}$ contains a τ-sheaf of rank ρ. Equivalently, this means that $\hat{\mathcal{M}}_\ell$ contains a smooth τ-sheaf $\hat{\mathcal{M}}'_\ell$ on $\hat{C}_{R,\ell}$ of rank ρ.

As explained in [TW] Section 6, there exists a covariant functor H_ℓ from the category of ℓ-adic τ-sheaves on $\hat{C}_{X,\ell}$ to the category of free $\mathbf{A}_\ell$-modules endowed with a continuous action by $\pi_1(\operatorname{Spec} X)$: to a τ-sheaf $\hat{M}_\ell$, one associates the scheme of $\mathbf{A}_\ell$-modules defined by

$$H_\ell(\hat{M}_\ell)(X') = (\hat{M}_\ell \hat{\otimes}_{\mathcal{O}_{C_{X,\ell}}} \mathcal{O}_{C_{X',\ell}})^{\tau=1} \tag{1}$$

for every X-scheme X'.

Following Drinfeld (cf. [TW] Proposition 6.2), the functor H_ℓ defines an *equivalence* between the category of smooth ℓ-adic τ-sheaves of rank r on $\hat{C}_{X,\ell}$ and the category of X-schemes of free $\mathbf{A}_\ell$-modules of rank r with continuous action. As a consequence, a smooth ℓ-adic τ-sheaf $\hat{N}_\ell$ on $\hat{C}_{K,\ell}$ extends to a smooth model $\hat{C}_{R,\ell}$ if and only the $\mathbf{A}_\ell[\Gamma_K]$-module $H_\ell(\hat{N}_\ell)$ of rank r is unramified.

For a τ-sheaf M on C_K, the module $H_\ell(M)$ is defined as $H_\ell(\hat{M}_\ell)(\operatorname{Spec} K^{\mathrm{sep}})$, where $\hat{M}_\ell$ is the completion of M along $\{\ell\} \times \operatorname{Spec} K$.

Theorems 1.1 and 1.2 can now be put into different words as follows:

Theorem 2.1 (Main theorem). *Let M be a τ-sheaf on C_K with characteristic ι and ℓ a closed point of C such that $\hat{M}_\ell$ is smooth. If $\hat{M}_\ell$ extends to a τ-sheaf $\hat{\mathcal{M}}'_\ell$ on $\hat{C}_{R,\ell}$ of nondegenerate rank ρ, then there exists a model $\mathcal{M}$ for M on C_R whose nondegenerate rank is at least ρ.*

The proof of Theorem 2.1 will take up to Section 5.

Proof of "Theorem 2.1 $\Rightarrow$ Theorems 1.1 and 1.2". Both for Theorems 1.1 and 1.2, the implications (i) $\Rightarrow$ (ii) are well known: cf. [Ga1] Proposition 5.1, resp. Proposition 5.9. As (ii) $\Rightarrow$ (iii) is completely trivial, we are left to prove the implication (iii) $\Rightarrow$ (i) of Theorem 1.2 (the case of good reduction Theorem 1.1 is dealt with by taking r' to be the rank of M in Theorem 1.2).

Let M be a τ-sheaf on C_K of rank r with characteristic $\imath$ and ℓ a closed point of $C\backslash\{\ell_x\}$. Suppose that $H_\ell(M)$ contains an unramified submodule H which is free over $\mathbf{A}_\ell$ of rank at least r'. By the above, this means that $\hat{M}_\ell$ contains a sub-τ-sheaf $\hat{N}_\ell$ that extends to a smooth ℓ-adic τ-sheaf $\hat{\mathcal{N}}_\ell$ on $\hat{C}_{R,\ell}$ of rank at least r'. Obviously, we can extend $\hat{\mathcal{N}}_\ell$ to a model $\hat{\mathcal{M}}'_\ell$ for $\hat{M}_\ell$ with nondegenerate rank at least r'. By Theorem 2.1, this implies that there exists a model $\mathcal{M}$ for M on C_R of nondegenerate rank at least r'. $\square$

2.2. *Sketch of the proof of the main theorem*

2.2.1. *Settings*

We want to give a sketch of the proof of Theorem 2.1, where, as a first approach, we assume that $C = \mathbb{A}^1$, that ℓ is a closed point of degree 1 of $\mathbb{A}^1$ and that $\hat{\mathcal{M}}'_\ell$ is smooth. In Section 5, we will show how to deal with the general case.

Let t denote a generator for the maximal ideal in $\mathbf{A}$ defining the point ℓ. Let ϖ be the generic point of the special fibre C_k. We can make the following identifications:

$$
\begin{aligned}
H^0(C_K, \mathcal{O}_{C_K}) &\cong K[t],\\
H^0(C_R, \mathcal{O}_{C_R,\varpi}) &\cong \mathcal{O}_\varpi := R[t]_{(\pi)},\\
H^0(\hat{C}_{K,\ell}, \mathcal{O}_{\hat{C}_{K,\ell}}) &\cong K[[t]],\\
H^0(\hat{C}_{R,\ell}, \mathcal{O}_{\hat{C}_{R,\ell}}) &\cong R[[t]]. \qquad (2)
\end{aligned}
$$

Let M be a τ-sheaf on $\mathbb{A}^1_K$ with characteristic $\imath$, and let us denote the $K[t]$-module of its global sections by M as well. We suppose the τ-module $\hat{M}_\ell$ over $K[[t]]$ extends to a smooth τ-module $\hat{\mathcal{M}}'_\ell$ over $R[[t]]$. The naive idea is to construct a good model $\mathcal{M}$ for M by putting

$$\mathcal{M} := M \cap \hat{\mathcal{M}}'_\ell \subset \hat{M}_\ell.$$

2.2.2. *Notations*

We call a monic polynomial

$$h(t) = t^d + \sum_{i=0}^{d-1} h_\nu t^\nu \in R[t]$$

strict if $v(h_v) > 0$ for all v. Every nonzero element $g \in R[[t]]$ has a unique decomposition

$$g = u \cdot \pi^{v_g} \cdot \tilde{g}$$

such that $u \in R[[t]]^{\times}$, $v_g \geqslant 0$ and where $\tilde{g}$ is a strict monic polynomial in $R[t]$ (Weierstraß preparation for $R[[t]]$). We have a valuation v_π on $R[[t]]$ given by $v_\pi(g) := v_g$, and extend this valuation to the quotient field Q of $R[[t]]$.

We consider the discrete valuation subring

$$B := \{g \in Q; v_\pi(g) \geqslant 0\} \subset Q.$$

Its residue field is isomorphic to $k((t))$. Finally, we set $\mathcal{O}_\varpi := R[t]_{(\pi)}$. By the unique factorization in $R[[t]]$, we have

$$\mathcal{O}_\varpi = K(t) \cap B \subset Q.$$

To keep track of all these rings, the following diagram might be useful; we have inclusions from left to right and from top to bottom.

$$\begin{tikzcd}
R[t] \arrow[rr, hook] \arrow[dr, hook] \arrow[dd, hook] & & R[t]_{(t,\pi)} \arrow[rr, hook] \arrow[dr, hook] \arrow[dd, hook] & & \mathcal{O}_\varpi \arrow[dr, hook] \arrow[dd, hook] & \\
& K[t] \arrow[rr, hook] \arrow[dd, hook] & & K[t]_{(t)} \arrow[rr, hook] \arrow[dd, hook] & & K(t) \arrow[dd, hook] \\
R[[t]] \arrow[rr, hook] \arrow[dr, hook] & & B \cap K[[t]] \arrow[rr, hook] \arrow[dr, hook] & & B \arrow[dr, hook] & \\
& K \otimes_R R[[t]] \arrow[rr, hook] & & Q \cap K[[t]] \arrow[rr, hook] \arrow[d, hook] & & Q \arrow[d, hook] \\
& & & K[[t]] \arrow[rr, hook] & & K((t))
\end{tikzcd}$$

We extend the Frobenius φ on R to a $\mathbb{F}_q$-linear endomorphism $\sigma := \varphi \otimes id$ on $R[t] = R \otimes \mathbb{F}_q[t]$. This induces in a unique way endomorphisms on all of the aforementioned rings.

2.2.3. Sketch

We proceed as follows:

(i) The module $\hat{\mathcal{M}}'_\ell$, an $R[[t]]$-submodule of $\hat{\mathcal{M}}_\ell$ is actually contained in

$$(K \otimes_R R[[t]]) \otimes_{K[t]} M \subset \hat{M}_\ell$$

(Proposition 3.1). In other words, with respect to a basis for $\hat{M}_\ell$ contained in M, the coordinates of elements of $\hat{\mathcal{M}}'_\ell$ have bounded denominator.

(ii) Denoting the stalk of M at the generic point of C_K by V, the $\mathcal{O}_\varpi$-module

$$\mathcal{M}_\varpi := V \cap (B \cdot \hat{\mathcal{M}}'_\ell) \subset \mathfrak{Q} := Q \otimes_{K(t)} V$$

is free and is of full rank inside V (Lemma 4.1).

(iii) Using the fact that $\hat{\mathscr{M}}'_\ell$ is smooth, we show that $\mathscr{M}_\varpi$ is a good τ-module over $\mathcal{O}_\varpi$ (Proposition 4.3).
(iv) The sheaf M together with $\mathscr{M}_\varpi$ define in a unique way (cf. [Ga1] Corollary 2.9) a good model $\mathscr{M}$ over C_R.

3. Models of ℓ-adic τ-sheaves

Proposition 3.1. *Let M be τ-sheaf on $\mathbb{A}^1_K$ with characteristic ι such that $\hat{M}_\ell$ is smooth at the closed point ℓ of degree 1 of C. Suppose the τ-sheaf $\hat{\mathscr{M}}'_\ell$ over $\hat{C}_{R,\ell}$ is an extension of $\hat{M}_\ell$. Then the module $\hat{\mathscr{M}}'_\ell$ is contained in*

$$(K \otimes_R R[[t]]) \otimes_{K[t]} M \subset \hat{M}_\ell.$$

Proof. (a) We want to apply a result of Anderson ([An2] Theorem 1). Let $\mathbf{m}$ be a $K[t]$-basis for M. As a locally free $R[[t]]$-module, $\hat{\mathscr{M}}'_\ell$ is actually free; let $\mathbf{q}$ be an $R[[t]]$-basis for $\hat{\mathscr{N}}_\ell \subset \hat{M}_\ell$. We express $\mathbf{q}$ in terms of the $K[[t]]$-basis $\mathbf{m}$ for $\hat{M}_\ell$ by means of a matrix $\Psi \in \mathrm{Mat}_{r\times r}(K[[t]])$:

$$\mathbf{q} = \mathbf{m} \cdot \Psi.$$

We have to show that $\Psi \in \mathrm{Mat}_{r\times r}(K \otimes R[[t]])$.

Further, we denote by $\Delta \in \mathrm{Mat}_{r\times r}(K[t])$ and $\hat{\Delta} \in \mathrm{Mat}_{r\times r}(R[[t]])$ the matrix representations of τ on the modules M and $\hat{\mathscr{M}}'_\ell$, respectively, i.e. $\tau(\mathbf{m}) = \mathbf{m} \cdot \Delta$ and $\tau(\mathbf{q}) = \mathbf{q} \cdot \hat{\Delta}$. These representations are related to each other by the equation

$$\Delta \cdot {}^\sigma \Psi = \Psi \cdot \hat{\Delta}. \tag{3}$$

(b) Let θ be given as the image of t under the map $\iota^* : \mathbf{A} \cong \mathbb{F}_q[t] \to K : t \mapsto \theta$ defining ι. As M has characteristic ι, we have $\det \Delta = h \cdot (t-\theta)^d$ for some constant $h \in K^\times$ and a positive integer d. The τ-sheaf $\hat{M}_\ell$ being smooth, we must have $\theta \neq 0$. Let $\tilde{\Delta}$ be the modified adjoint matrix in $\mathrm{Mat}_{r\times r}$ satisfying

$$\tilde{\Delta} \cdot \Delta = (t-\theta)^d.$$

Upon multiplying both sides of Eq. (3) by $\tilde{\Delta}$, we get:

$$(t-\theta)^d \cdot {}^\sigma \Psi = \tilde{\Delta} \cdot \Psi \cdot \hat{\Delta}.$$

(c) The equation ${}^\sigma z = (t-\theta) \cdot z$ has a nonzero solution $c \in K^{\mathrm{sep}}[[t]]$, as one checks immediately (it is a nonzero element of the Tate module of the Carlitz module, cf. [Go1] Chapter 3). The matrix $\Psi' := c^d \cdot \Psi$ then satisfies the equation

$${}^\sigma Z = \tilde{\Delta} \cdot Z \cdot \hat{\Delta} \tag{4}$$

for $Z \in \mathrm{Mat}_{r\times r}(\bar{K}[[t]])$. We claim that every solution Z for this equation is contained in $\mathrm{Mat}_{r\times r}(K\otimes\bar{R}[[t]])$. Let us write out $Z := \sum_{i=1}^{\infty} Z_i t^i$, introducing matrices $Z_i = (Z_i)_{kl} \in \mathrm{Mat}_{r\times r}(\bar{K})$. For all $i \geqslant 0$, we set

$$v(Z_i) := \min_{k,l}\{v((Z_i)_{kl})\} \in \mathbb{Z}\cup\{+\infty\}.$$

For $\tilde{\Delta} = (\tilde{\Delta}_{k,l})_{kl}$, considered as a matrix in $\mathrm{Mat}_{r\times r}(K\otimes R[[t]])$, we put

$$v(\tilde{\Delta}) := \min_{k,l}\ \{v(\tilde{\Delta}_{kl})\};$$

we do the same for $\hat{\Delta}$.

Comparing the coefficients of t^n in Eq. (4), we get

$${}^{\sigma}Z_n = \sum_{i+j+k=n} \tilde{\Delta}_i \cdot Z_j \cdot \hat{\Delta}_k.$$

Thus, we see that

$$q\cdot v(Z_n) \geqslant v(\tilde{\Delta}) + v(\hat{\Delta}) + \min_{j\leqslant n}\ v(Z_j),$$

and it follows by induction on n that

$$(q-1)v(Z_n) \geqslant v(\tilde{\Delta}) + v(\hat{\Delta}).$$

This shows that $\Psi' \in \mathrm{Mat}_{r\times r}(K\otimes\bar{R}[[t]])$ indeed.

(d) We now distinguish two cases:

(i) $\boxed{v(\theta)\leqslant 0}$. Putting $\zeta := \theta^{-1} \in R$, we rewrite $(t-\theta)$ as

$$-\zeta^{-1}\cdot(1-\zeta\cdot t).$$

The power series $c^{-1} \in \bar{K}[[t]]$ satisfies the equation

$$z = -\zeta^{-1}(1-\zeta\cdot t)\cdot {}^{\sigma}z.$$

An easy calculation shows that the solutions for this equations are contained in $K\otimes\bar{R}[[t]]$. Therefore, the matrix

$$\Psi = c^{-d}\cdot\Psi'$$

is a matrix with coefficients in $K\otimes\bar{R}[[t]]$, and, as it was defined over $K[[t]]$, we may conclude that

$$\Psi \in \mathrm{Mat}_{r\times r}(K\otimes R[[t]]).$$

(ii) $\boxed{v(\theta)>0}$. We are now in the situation of [An2] Theorem 1, p. 52. Consider the matrix $\Psi = c^{-d}\Psi'$, seen as a matrix whose entries are meromorphic functions the open unit disk

$$D^0_{\bar{K}} := \{t\in\bar{K};\, v(t)<1\}$$

(viewed as a rigid analytic space). Following Anderson, one first proves that Ψ has no poles (working over the completion $\mathbf{C}$ of $\bar{K}$, to be more precise). Next, using the fact that Ψ' has entries in $K\otimes\bar{R}[[t]]$

and some estimates on c, one deduces that Ψ has entries in $K \otimes R[[t]]$ indeed. $\square$

4. Nondegenerate formal τ-modules

Let V be an r-dimensional $K(t)$-vector space and put

$$\mathfrak{Q} := Q \otimes_{K(t)} V.$$

For a given free B-submodule $\mathfrak{B}$ of $\mathfrak{Q}$ of rank r, we define the $\mathcal{O}_\varpi$-module

$$\mathcal{M}_\varpi := V \cap \mathfrak{B}.$$

Lemma 4.1. *Using the above assumptions, the $\mathcal{O}_\varpi$-module $\mathcal{M}_\varpi := V \cap \mathfrak{B}$ is free of rank r and the cokernel of $B \cdot \mathcal{M}_\varpi \to \mathfrak{B}$ has finite length as a B-module.*

$$\begin{tikzcd} \mathcal{O}_\varpi \arrow[r, hook] \arrow[d, hook] & K(t) \arrow[d, hook] \\ B \arrow[r, hook] & Q \end{tikzcd} \qquad \begin{tikzcd} \mathcal{M}_\varpi \arrow[r, hook] \arrow[d, hook] & V \arrow[d, hook] \\ \mathfrak{B} \arrow[r, hook] & \mathfrak{Q} \end{tikzcd}$$

Proof. Let us choose a $K(t)$-basis $\mathbf{v} := (v_1, \dots, v_r)$ for V. We also fix a B-basis $\mathbf{b} := (b_1, \dots, b_r)$ for the free module $\mathfrak{B}$ and express $\mathbf{v}$ in terms of $\mathbf{b}$ by means of a matrix $\Omega \in \mathrm{Mat}_{r\times r}(Q)$ as follows: $\mathbf{v} = \mathbf{b} \cdot \Omega$. After dividing $\mathbf{v}$ by a suitable power of π, we can assume that Ω has coefficients in B, so that the elements v_i are contained in $\mathcal{M}_\varpi$.

There exists an $\omega \in \mathbb{Z}$ such that $\pi^\omega \cdot \Omega^{-1}$ has entries in B. Let us write any element $n \in \mathcal{M}_\varpi$, as $n = \mathbf{b} \cdot \Lambda$ with $\Lambda \in \mathrm{Mat}_{r\times 1}(B)$. It then follows that

$$\pi^\omega n = \pi^\omega \mathbf{b} \cdot \Lambda = \mathbf{v} \cdot (\pi^\omega \Omega^{-1}) \cdot \Lambda \subset \mathbf{v} \cdot \mathrm{Mat}_{r\times 1}(B). \tag{5}$$

This shows that $\pi^\omega \mathcal{M}_\varpi$ is contained in the $\mathcal{O}_\varpi$-module generated by $\mathbf{v}$. As $\mathcal{O}_\varpi$ is a noetherian principal ideal domain, the torsion free module $\mathcal{M}_\varpi$ is therefore finitely generated and hence free of rank r. This implies that the cokernel of

$$B \cdot \mathcal{M}_\varpi \to \mathfrak{B}$$

is a torsion module. $\square$

Remark 4.2. The following example shows that $B \cdot \mathcal{M}_\varpi$ may be strictly smaller than $\mathfrak{B}$. Let α be an element in B with reduction $\bar{\alpha} \in k((t))$. Consider the two-dimensional $K(t)$-vector space spanned by a basis $\mathbf{v} = (v_1, v_2)$, and the B-module $\mathfrak{B} \subset \mathfrak{Q}$ generated by $\mathbf{b} = (\hat{n}_1, \hat{n}_2)$ such that

$$\mathbf{v} = \mathbf{b} \cdot \begin{pmatrix} 1 & \pi^{-1}\alpha \\ 0 & \pi^{-1} \end{pmatrix}.$$

It is easy to see that, if $\bar{\alpha} \notin k(t)$, then $V \cap \mathfrak{B}$ is generated by $\mathbf{v}$, such that we have an exact sequence of B-modules

$$0 \to B \cdot \mathscr{M}_{\varpi} \to \mathfrak{B} \to B/(\pi) \to 0.$$

Using the above assumptions, let us further suppose that V is a τ-module over $K(t)$ and that $\mathfrak{B}$ is a τ-invariant submodule of $\mathfrak{Q}$.

Proposition 4.3. *The τ-modules $\mathfrak{B}$ over B and $\mathscr{M}_{\varpi}$ over $\mathcal{O}_{\varpi}$ have the same nondegenerate rank.*

Proof. (a) By the previous lemma, $\mathscr{M}_{\varpi}$ has the same rank as $\mathfrak{B}$. Let $\bar{\mathscr{M}}'_{\varpi}$ denote the reduction of $\mathscr{M}_{\varpi}$. As k is perfect, we have an exact sequence of $k(t)[\tau]$-modules as follows (cf. [Ga1] Remark 2.6):

$$0 \to (\bar{\mathscr{M}}'_{\varpi})_1 \to \bar{\mathscr{M}}'_{\varpi} \to (\bar{\mathscr{M}}'_{\varpi})_{\mathrm{nil}} \to 0,$$

where $(\bar{\mathscr{M}}'_{\varpi})_1$ is a τ-module (whose rank we will denote by r'), whereas the action of τ on $(\bar{\mathscr{M}}'_{\varpi})_{\mathrm{nil}}$ is nilpotent. We choose a $k(t)$-basis $(\bar{n}_1, \dots, \bar{n}_{r'})$ (resp. $(\bar{n}_{r'+1}, \dots, \bar{n}_r)$) for $(\bar{\mathscr{M}}'_{\varpi})_1$ (resp. $(\bar{\mathscr{M}}'_{\varpi})_{\mathrm{nil}}$). Finally, we fix a lift

$$\mathbf{n} := (n_1, \dots, n_{r'}; n_{r'+1}, \dots, n_r)$$

for $(\bar{n}_1, \dots, \bar{n}_r)$ in $\mathscr{M}_{\varpi}$, which yields an $\mathcal{O}_{\varpi}$-basis for $\mathscr{M}_{\varpi}$, and, for every $s > 0$, we denote by $\Delta_s \in \mathrm{Mat}_{r \times r}(\mathcal{O}_{\varpi})$ the matrix representation of τ relatively to the basis $\mathbf{n}$, i.e. $\tau^s(\mathbf{n}) = \mathbf{n} \cdot \Delta_s$.

(b) We have a similar filtration of the $k((t))[\tau]$-module $\bar{\mathfrak{B}}$ yielding modules $\bar{\mathfrak{B}}_1$ and $\bar{\mathfrak{B}}_{\mathrm{nil}}$. Note that $(\bar{\mathscr{M}}'_{\varpi})_1$ injects into $\bar{\mathfrak{B}}_1$ (same argument as in Lemma 8.3), which yields that $r' \leqslant \rho$.

We now assume that $r' < \rho$, where ρ is the nondegenerate rank of $\mathfrak{B}$, and want to deduce a contradiction. Let us extend $(\bar{n}_1, \dots, \bar{n}_{r'})$ to a $k((t))$-basis

$$\bar{\mathbf{b}}_1 := (\bar{n}_1, \dots, \bar{n}_{r'}; \bar{b}_{r'+1}, \dots, \bar{b}_{\rho})$$

for $\bar{\mathfrak{B}}_1$. For all $s > 0$, let $(\bar{\Delta}_1)_s \in \mathrm{GL}_{\rho}(k((t)))$ denote the matrix representation the action of τ^s on $\bar{\mathfrak{B}}_1$ with respect to this basis $\bar{\mathbf{b}}_1$: $\tau^s(\bar{\mathbf{b}}_1) = \bar{\mathbf{b}}_1 \cdot (\bar{\Delta}_1)_s$.

(c) On the other hand, we choose a $k((t))$-basis $(\bar{b}_{\rho+1}, \dots, \bar{b}_r)$ for $\bar{\mathfrak{B}}_{\mathrm{nil}}$. Taking some lift $(b_{r'+1}, \dots b_r)$ of $(\bar{b}_{r'+1}, \dots \bar{b}_r)$ to $\mathfrak{B}$, we obtain a B-basis

$$\mathbf{b} = (n_1, \dots, n_{r'}; b_{r'+1}, \dots, b_{\rho}; b_{\rho+1}, \dots b_r)$$

for $\mathfrak{B}$. We denote the B-module spanned by the elements $n_1, \dots, n_{r'}$ by $\mathfrak{B}_0$, and further put $\mathfrak{B}_\star := \langle b_{r'+1}, \dots, b_{\rho} \rangle$ and $\mathfrak{B}_2 := \langle b_{\rho+1}, \dots b_r \rangle$.

Notice that a different choice of the elements $(\bar{b}_{r'+1}, \dots, \bar{b}_{\rho}; \bar{b}_{\rho+1}, \dots, \bar{b}_r)$ or their respective lifts would correspond to a basis transformation $\mathbf{b}' := \mathbf{b} \cdot U$

for a matrix in

$$\mathcal{U} := \left\{ \left(\begin{array}{c|c|c} 1 & * & * \\ \hline 0 & * & * \\ \hline 0 & \pi * & * \end{array} \right) \right\} \subset \mathrm{GL}_r(B),$$

where the blocks correspond to the composition $\mathfrak{B} = \mathfrak{B}_1 \oplus \mathfrak{B}_2 \oplus \mathfrak{B}_3$.

(d) We express $\mathbf{n}$ in terms of $\mathbf{b}$ by means of a matrix

$$\Omega \in \mathrm{GL}_r(Q) \cap \mathrm{Mat}_{r\times r}(B)$$

as follows: $\mathbf{n} = \mathbf{b} \cdot \Omega$. The matrix Ω is then of the form

$$\left\{ \left(\begin{array}{c|c|c} 1 & * & * \\ \hline 0 & * & * \\ \hline 0 & * & * \end{array} \right) \right\}.$$

Note that taking a different choice of $\mathbf{b}$ amounts to replacing Ω by $U \cdot \Omega$, for some $U \in \mathcal{U}$. We can find a basis $\mathbf{b}$ such that Ω (and more precisely its $r' + 1$st column) is of the form:

$$\Omega = \begin{array}{c} \quad \overset{r'+1}{\downarrow} \\ \left(\begin{array}{c|c|c|c} 1 & Z & * & * \\ \hline 0 & \pi^c & * & * \\ \hline 0 & 0 & * & * \\ \hline 0 & W & * & * \end{array} \right) \end{array},$$

where $X \in \mathrm{Mat}_{r'\times 1}(B)$, $c \geqslant 0$ and

$$W \in \pi^c \cdot \mathrm{Mat}_{(r-\rho)\times 1}(B).$$

The blocks correspond to the decomposition $\mathfrak{B} = \mathfrak{B}_0 \oplus \mathfrak{B}_{\bigstar,1} \oplus \mathfrak{B}_{\bigstar,2} \oplus \mathfrak{B}_2$, where we put $\mathfrak{B}_{\bigstar,1} := \langle b_{r'+1} \rangle$ and $\mathfrak{B}_{\bigstar,2} := \langle b_{r'+1}, \ldots, b_\rho \rangle$; we consider an analogous decomposition for $\mathcal{M}_\varpi$.

(e) For any integer s, we will denote by $\hat{\Delta}_s \in \mathrm{Mat}_{r\times r}(B)$ the matrix representing τ^s with respect to $\mathbf{b} : \tau^s(\mathbf{b}) = \mathbf{b} \cdot \hat{\Delta}_s$. Observe that the representations $\hat{\Delta}_s$ and Δ_s satisfy the following relation:

$$\hat{\Delta}_s \cdot {}^{\sigma^s}\Omega = \Omega \cdot \Delta_s. \tag{6}$$

We distinguish two cases:

(1) Suppose $c > 0$. For $s > 0$, let us write $\hat{\Delta}_s$ as a block matrix of matrices $(\delta \mid \delta' \mid \delta'' \mid \delta''')$ with respect to the decomposition of $\mathfrak{B}$; we do the same for $\Delta_s = (d \mid d' \mid d'' \mid d''')$. Eq. (6) can then be rewritten as

$$(\delta \mid \delta' \mid \delta'' \mid \delta''') \cdot \left(\begin{array}{c|c|c|c} 1 & {}^{\sigma^s}Z & * & * \\ \hline 0 & \pi^{q^s c} & * & * \\ \hline 0 & 0 & * & * \\ \hline 0 & {}^{\sigma^s}W & * & * \end{array} \right) = \Omega \cdot (d \mid d' \mid d'' \mid d'''). \tag{7}$$

We thus obtain the equations

$$\begin{cases} \delta & = \Omega d, \\ \delta^{\sigma^s} Z + \pi^{q^s c}(\delta' + \delta'''^{\sigma^s} W) & = \Omega d'. \end{cases} \tag{8}$$

Assume that s is large enough so that $v_\pi(\det \Omega) < q^s c$, which implies that

$$\pi^{q^s c} \cdot \Omega^{-1} \in \pi \cdot \mathrm{Mat}_{r \times r}(B).$$

It then follows from (8) that

$$d^{\sigma^s} Z + (\pi^{q^s c} \cdot \Omega^{-1})(\delta' + \delta'''^{\sigma^s} W) = d'. \tag{9}$$

We denote by $\bar{d} \in \mathrm{Mat}_{r \times r'}(k(t))$ the reduction of d modulo π; similarly, we define $\bar{d'}$. Also, we consider the reductions

$$\bar{\delta} \in \mathrm{Mat}_{r \times r'}(k((t)))$$

and $\bar{Z} \in \mathrm{Mat}_{\rho \times 1}(k((t)))$ of δ and Z, respectively. Reducing mod π, Eq. (9) gives

$$\bar{d}^{\sigma^s} \bar{X} = \bar{d'}. \tag{10}$$

As $\bar{\delta}$ has full rank r', we deduce from $\bar{\delta} = \bar{\Omega} \cdot \bar{d}$ that $\bar{d}$ has full rank r', too. Therefore, the solution $\bar{Z}$ of (10) is unique, namely $\bar{Z}$, and must therefore be algebraic, i.e.

$$\bar{Z} \in \mathrm{Mat}_{\rho \times 1}(k(t)),$$

as $\bar{d}$ and $\bar{d'}$ have entries in $k(t)$ as well. Finally, let us take the canonical lift Z_0 of $\bar{Z}$ to $\mathcal{O}_\varpi$ and put $Z = Z_0 + \pi Z_1$, for $Z_1 \in \mathrm{Mat}_{r' \times 1}(B)$. The element

$$v := \pi^{-1}\mathbf{n} \cdot \begin{pmatrix} -Z_0 \\ \hline 1 \\ \hline 0 \\ \hline 0 \end{pmatrix} = \mathbf{b} \cdot \begin{pmatrix} Z_1 \\ \hline \pi^{c-1} \\ \hline 0 \\ \hline \pi^{-1} W \end{pmatrix}$$

is contained both in V and in $\mathfrak{B}$, but not contained in $\mathcal{M}_\varpi$, which obviously contradicts the definition of $\mathcal{M}_\varpi$.

(2) If $c = 0$, then we see, denoting by $\bar{\Omega}$ the reduction of Ω modulo π, that the upper left $(r'+1) \times (r'+1)$-block of $\bar{\Omega}$ has full rank $r'+1$. Also, the upper left $\rho \times \rho$ block of $\bar{\Delta}_s$, namely $(\bar{\Delta}_1)_s$ has full rank ρ. We see from this that the matrix

$$\bar{\Delta}_s \cdot {}^{\sigma^s}\bar{\Omega} = \bar{\Omega} \cdot \bar{\Delta}_s$$

has rank at least $r'+1$, for all $s > 0$. In particular, the nondegenerate rank of Δ_s is at least $r'+1$, which gives a contradiction. $\square$

5. Proof of the main theorem

Proof of Theorem 2.1. (a) First of all, we can reduce ourselves to the case that $C = \mathbb{A}^1$. Indeed, for a general curve C, we consider a finite morphism

$$f : C \to \mathbb{A}^1,$$

and denote the induced morphism $C_K \to \mathbb{A}^1_K$ (resp. $C_R \to \mathbb{A}^1_R$) again by f. If $\mathcal{M}^\star$ is the maximal model for $f_*(M)$ on $\mathbb{A}^1_R$ and $\mathcal{M}$ the maximal model for M on C_R, then $f_*(\mathcal{M}) = \mathcal{M}^\star$.

Let ℓ be a closed point of C. If $\hat{\mathcal{M}}'_\ell$ has nondegenerate rank ρ on $\hat{C}_{R,\ell}$, then $f_*(\hat{\mathcal{M}}'_\ell)$ has nondegenerate rank at least $\deg f \cdot \rho$. Assuming that the theorem holds for $\mathbb{A}^1$, we then obtain a model with nondegenerate rank at least $\deg f \cdot \rho$. The maximal model $\mathcal{M}^\star$ has at least the same nondegenerate rank. This in turn implies that $\mathcal{M}$ has nondegenerate rank at least ρ.

(b) Next, we show that it suffices to prove the result over a finite étale extension R' of R. We can assume that R'/R is a Galois. Let ϖ' denote the generic point of the special fibre of $C_{R'}$ and $\hat{\mathcal{O}}_{\varpi'}$ the completion of $\mathcal{O}_{C_{R'},\varpi'}$ with respect to its maximal ideal.

Suppose that $\hat{\mathcal{M}}'_\ell$ has nondegenerate rank ρ on $\hat{C}_{R,\ell}$. By assumption, the maximal model $\mathcal{M}'$ on $C_{R'}$ of $M_{K'} := M \otimes_K K'$ has nondegenerate rank r' at least ρ. Let $\hat{\mathcal{M}}'_{\varpi'}$ be the completion of the stalk of $\mathcal{M}'$ at ϖ'. In [Ga1] Proof of Proposition 4.6, we calculated how this nondegenerate reduction can be lifted to a maximal good sub-τ-module $\hat{\mathcal{M}}'_1$ of $\hat{\mathcal{M}}'_{\varpi'}$ of rank r' over $\hat{\mathcal{O}}_{\varpi'}$. As $\hat{\mathcal{M}}'_1$ is maximal, it is Galois invariant.

By Galois descent, the sheaf $\hat{\mathcal{M}}'_1$ descends to a sheaf $\hat{\mathcal{M}}_1$ on $\hat{\mathcal{O}}_\varpi$ and the morphism

$$\tau : \sigma^* \hat{\mathcal{M}}'_1 \to \hat{\mathcal{M}}'_1$$

yields a morphism $\tau : \sigma^* \hat{\mathcal{M}}_1 \to \hat{\mathcal{M}}_1$. As properties of modules and morphisms descend as well, we obtain that this morphism τ is an isomorphism on $\hat{\mathcal{M}}_1$ (in particular, $\hat{\mathcal{M}}_1$ a τ-module over $\hat{\mathcal{O}}_\varpi$). This shows that the nondegenerate rank of the maximal model $\mathcal{M}$ of M on C_R is at least r'.

(c) We now assume that $C = \mathbb{A}^1$. Let ℓ be a closed point of $\mathbb{A}^1$ of degree s, with residue field κ_ℓ. By (b), we may assume the finite field κ_ℓ injects into R. Let

$$\{\ell'_1, \ell'_2, \ldots, \ell'_s\}$$

be the set of closed points of lying above ℓ on $\mathbb{A}^1_{\kappa_\ell}$. Let $\hat{C}_{\kappa_\ell,\ell'_i}$ denote the completion of $\mathbb{A}^1_{\kappa_\ell}$ at ℓ'_i, and put

$$\mathbf{A}'_{\ell'_i} := \mathcal{O}_{\hat{C}_{\kappa_\ell,\ell'_i}}.$$

As C is smooth, we have $\mathbf{A}'_{\ell'_i} \cong \kappa_\ell[[\lambda_i]]$, where λ_i denotes a uniformizer at the point ℓ'_i. The ring $R \hat{\otimes}_{\kappa_\ell} \mathbf{A}'_{\ell'_i}$ is then isomorphic to $R[[\lambda_i]]$. Finally

$$\mathcal{O}_{\hat{C}_{R,\ell}} = R \ \hat{\otimes}_{\mathbb{F}_q} \mathbf{A}_\ell \cong \prod_i R \hat{\otimes}_{\kappa_\ell} \mathbf{A}'_{\ell'_i} = \prod_i R[[\lambda_i]].$$

Similarly,

$$\mathcal{O}_{\hat{C}_{K,\ell}} \cong \prod_i K[[\lambda_i]].$$

The endomorphism σ of $\mathcal{O}_{\hat{C}_{R,\ell}}$ induces morphisms $\sigma : R[[\lambda_i]] \to R[[\lambda_j]]$ for all pairs (i,j) such that ${}^{\sigma}\ell_i = \ell_j$.

Any $\mathcal{O}_{\hat{C}_{R,\ell}}$-module $\hat{\mathcal{M}}'_\ell$ can be written as a product $\hat{\mathcal{M}}'_\ell = \prod_i \hat{\mathcal{N}}_i$, where the $\hat{\mathcal{N}}_i$ are $R[[\lambda_i]]$-modules. If $\hat{\mathcal{M}}'_\ell$ is endowed with a σ-linear endomorphism

$$\tau : \sigma^* \hat{\mathcal{M}}'_\ell \to \hat{\mathcal{M}}'_\ell$$

then τ will induce morphisms

$$\tau : \sigma^* \hat{\mathcal{N}}_i \to \hat{\mathcal{N}}_j$$

if (i,j) satisfies ${}^{\sigma}\ell_i = \ell_j$ (the same applies to $\mathcal{O}_{\hat{C}_{K,\ell}}$-modules). Each of the modules $\mathcal{N}_i$ is τ^s-invariant.

(d) Let M and $\hat{\mathcal{M}}'_\ell$ be as in the statement of this theorem. Let us denote the stalk of M at the generic point of $\mathbb{A}^1_K$ by V. Let Q_i, B_i etc. denote the subrings of $K[[\lambda_i]]$ which we defined for $K[[t]]$ in Section 2.

We consider M as a τ^s-sheaf over $\mathbb{A}^1_K = \mathbb{A}^1_{K_\ell} \otimes_{K_\ell} \operatorname{Spec} K$. Now, upon replacing φ by φ^s, and τ by τ^s, we can apply Proposition 3.1 to the module $\hat{\mathcal{N}}_i \subset \hat{M}_i$ to obtain that

$$\hat{\mathcal{N}}_i \subset (K \otimes R[[\lambda_i]]) \otimes_{K[\lambda_i]} M.$$

We remark that $\mathcal{O}_\varpi = \mathcal{O}_{C_R,\varpi}$, and define the $\mathcal{O}_\varpi$-modules

$$(\mathcal{M}_\varpi)_i := V \cap (B_i \cdot \hat{\mathcal{N}}_i) \subset (Q_i)_\lambda \otimes_{K(\lambda_i)} V.$$

Again replacing τ by τ^s, we then deduce from Proposition 4.3 that the $(\mathcal{M}_\varpi)_i$ have the same nondegenerate rank as $\hat{\mathcal{M}}'_\ell$. Finally, the $\mathcal{O}_{C_R,\varpi}$-module

$$\mathcal{M}_\varpi := \sum_i (\mathcal{M}_\varpi)_i$$

is τ-invariant and nondegenerate of rank ρ. The stalk $\mathcal{M}_\varpi$ at ϖ, together with the generic fibre M, define a unique model $\mathcal{M}$ on C_R with the desired property, by [Ga1] Corollary 2.9. $\square$

6. Analytic semistability

Consider a τ-sheaf $\tilde{M}$ on the curve $C \times_{\mathbb{F}_q} \operatorname{Spec} K$, viewed as a rigid analytic space and denoted by $\tilde{C}_K$. We also fix an R-model $\tilde{C}_R$ for $\tilde{C}_K$: a formal scheme which is admissible in the sense of [BL] (i.e. its structural sheaf consists of entire functions) and which has generic fibre $\tilde{C}_K$. It is now merely a formal matter (along the lines of the techniques in [Ga1] Section 3) to reformulate and prove Theorem 1.2 for such analytic τ-sheaves.

In [Ga1] Definition 4.4, we called $\tilde{M}$ *semistable* over K if there exists

- a nonempty open subscheme $C' \subset C$ and
- a filtration
$$0 = \tilde{N}_0 \subset \tilde{N}_1 \subset \cdots \subset \tilde{N}_n = \tilde{M}|_{\tilde{C}'_K} \tag{11}$$
of $\tilde{M}|_{\tilde{C}'_K}$ by analytic τ-sheaves on $\tilde{C}'_K$ which are saturated in $\tilde{M}|_{\tilde{C}'_K}$, i.e. whose quotient sheaves $\tilde{M}|_{\tilde{C}'_K}/\tilde{N}_i$ are locally free, such that the subquotient τ-sheaves $\tilde{M}_i := \tilde{N}_i/\tilde{N}_{i-1}$ possess good models over $\tilde{C}'_R$.

Assume that the residue field k of R is algebraic over $\mathbb{F}_q$. In [Ga1] Theorem 4.11, we proved that then every analytic τ-sheaf is semistable over some finite extension K' of K. One now immediately obtains the following result as a corollary from Theorem 1.2:

Corollary 6.1 (Galois criterion for semistability). *Assume that the residue field k of R is algebraic over $\mathbb{F}_q$. Let $\tilde{M}$ be an analytic τ-sheaf on C_K of rank r with characteristic ι. Then the following statements are equivalent*:

(i) *The analytic τ-sheaf is semistable over K.*
(ii) *For all but a finite number of closed points ℓ of C, the action of the inertia group I_K of K on the $\mathbf{A}_\ell[\Gamma_K]$-module $H_\ell(\tilde{M})$ is unipotent.*
(iii) *There exists a closed point ℓ of C such that the $\mathbf{A}_\ell[\Gamma_K]$-module $H_\ell(\tilde{M})$ has $\mathbf{A}_\ell$-rank r and such that the action of I_K on $H_\ell(\tilde{M})$ is unipotent.*

Proof. The implication (i) $\Rightarrow$ (ii) is well known (cf. [Ga1] Theorem 5.2), and (ii) $\Rightarrow$ (iii) is trivial. For the proof (iii) $\Rightarrow$ (i) we can refer to that of [Ga1] Theorem 4.11. It suffices to remark, that by Theorem 1.2, the unipotency of the action of inertia implies that for each quotient of $\tilde{M}$ there exists a nondegenerate model over C'_K. $\square$

7. Local factors of L-functions

We now show how, for a given τ-sheaf M, the *local L-factor*, defined via the action of Frobenius on its Galois module $H_\ell(M)$, can be defined in terms of the reduced τ-sheaf $\bar{\mathscr{M}}$ and that, as a consequence, this factor has coefficients in $\mathbf{A}$ and is independent of the closed point ℓ, for almost all closed points ℓ of C.

Let R be a complete discrete valuation $\mathbb{F}_q$-algebra with fraction field K and finite residue field k. Let

$$d_x := [k : \mathbb{F}_q]$$

denote the degree of the closed point x of R. Let M be a τ-sheaf over C_K, $\mathscr{M}$ its maximal model over C_R and $\bar{\mathscr{M}}$ the reduction of $\mathscr{M}$ at x. As k is perfect, we have an exact sequence of sheaves on C_k with an action of τ as follows (cf. [Ga1] Remark 2.6):

$$0 \to \bar{\mathscr{M}}_1 \to \bar{\mathscr{M}} \to \bar{\mathscr{M}}_{\mathrm{nil}} \to 0, \tag{12}$$

where $\bar{\mathscr{M}}_1$ is a τ-sheaf on C_k (maximal amongst all τ-sheaves contained in $\bar{\mathscr{M}}$) whose rank we will denote by r', whereas the action of τ on $\bar{\mathscr{M}}_{\mathrm{nil}}$ is nilpotent.

For a closed point ℓ of $C \backslash \{\ell_x\}$, let

$$H_\ell(M)^{I_K}$$

be the $\mathbf{A}_\ell$-module of invariants of $H_\ell(M)$ under the action of I_K; it is a Γ_K-invariant direct summand of $H_\ell(M)$. The action of $\Gamma_k \cong \Gamma_K / I_K$ on $H_\ell(M)^{I_K}$ is well-defined; thus there is also a well-defined action of its canonical generator Frob_x, which acts as φ^{d_x} on k, and of its inverse, the *geometric Frobenius*.

In close analogy with the theory of L-functions associated with schemes of finite type over $\mathbb{Z}$ (cf. [Go2] Section 2), one defines the local L-factor for $H_\ell(M)$ at x as follows:

Definition 7.1 (Goss). The *local L-factor for $H_\ell(M)$ at x* is defined by

$$L_x(H_\ell(M); Z)^{-1} := \det_{\mathbf{A}_\ell}(1 - Z^{d_x} \cdot \mathrm{Frob}_x^{-1} \mid H_\ell(M)^{I_K}) \in \mathbf{A}_\ell[Z].$$

Following [Bö] Definition 1.40, we can also local L-factor to the reduction $\bar{\mathscr{M}}$ of a τ-sheaf $\mathscr{M}$ on C_{R_x}. In fact, the following Definition 7.2 makes sense for every coherent sheaf $\mathscr{M}$ on C_{R_x} endowed with a morphism $\tau : \sigma^*\mathscr{M} \to \mathscr{M}$, and we should note that it is this weaker definition of a τ-sheaf that is used in [Bö]:

Definition 7.2 (Böckle). We define the *local L-factor for M at x* by

$$L_x(M; Z)^{-1} = \det_{\mathbf{A}} (1 - Z \cdot \tau \mid H^0(C_k, \bar{\mathscr{M}})) \in \mathbf{A}[Z],$$

which, as was remarked in [Bö] Remark 1.41, also equals

$$\det_{\mathbf{A} \otimes k} (1 - Z^{d_x} \cdot \tau^{d_x} \mid H^0(C_k, \bar{\mathscr{M}})).$$

Theorem 7.3. *Let M be a τ-sheaf over C_K. For all but a finite number of closed points ℓ of C, we have*

$$L_x(M; Z) = L_x(H_\ell(M); Z).$$

In particular, the polynomial $L_x(H_\ell(M);Z)^{-1}$ *has coefficients in* $\mathbf{A}$ *and is independent of* ℓ *for all but a finite number of* ℓ.

Proof. (a) By a weak version of the analytic lifting theorem (cf. [Ga1] Remark. 4.9), we can lift $\bar{\mathscr{M}}_1$ to a saturated ℓ-adic sub-τ-sheaf

$$(\hat{\mathscr{M}}_\ell)_1 \subset \hat{\mathscr{M}}_\ell$$

on $\hat{C}'_{R,\ell}$ of rank r', for all the closed points ℓ of a nonempty open subscheme C' of C.

Clearly, the reduction $\overline{(\hat{\mathscr{M}}_\ell)_1}$ of $(\hat{\mathscr{M}}_\ell)_1$ satisfies

$$\overline{(\hat{\mathscr{M}}_\ell)_1} = \mathcal{O}_{\hat{C}_{k,\ell}} \otimes_{\mathcal{O}_{C_k}} (\bar{\mathscr{M}})_1.$$

Therefore, the ℓ-adic τ-sheaf $(\hat{\mathscr{M}}_\ell)_1$ is smooth outside a finite set S' of closed points of C'. We deduce for all closed points $\ell \in C'' \coloneqq C \backslash S'$:

$$\begin{aligned} L_x(M;Z)^{-1} &= \det_{\mathbf{A}\otimes k}(1 - Z^{d_x}\cdot\tau^{d_x} \mid \bar{\mathscr{M}}) = \det_{\mathbf{A}\otimes k}(1 - Z^{d_x}\cdot\tau^{d_x} \mid (\bar{\mathscr{M}})_1) \\ &= \det_{\mathbf{A}_\ell\otimes k}(1 - Z^{d_x}\cdot\tau^{d_x} \mid \mathcal{O}_{\hat{C}_{k,\ell}} \otimes_{\mathcal{O}_{C_k}} (\bar{\mathscr{M}})_1) \\ &= \det_{\mathbf{A}_\ell\otimes k}(1 - Z^{d_x}\cdot\tau^{d_x} \mid \overline{(\hat{\mathscr{M}}_\ell)_1}) \end{aligned}$$

(b) Let ρ be the rank of $H_\ell(M)^{I_K}$. By the equivalence discussed in Section 2, we deduce the existence of a maximal smooth ℓ-adic τ-sheaf $\hat{\mathscr{N}}_\ell$ of $\hat{\mathscr{M}}_\ell$ on $\hat{C}_{R,\ell}$ of rank ρ. Let $\bar{\mathscr{N}}_\ell$ denote the reduction of $\hat{\mathscr{N}}_\ell$ to $\hat{C}_{k,\ell}$ and T the linear endomorphism τ^{d_x} of $\bar{\mathscr{N}}_\ell$ on $\hat{C}_{k,\ell}$.

By the definition of H_ℓ (cf. (1)), τ^{d_x} operates as the identity on $H_\ell(M)^{I_K} = H_\ell(\bar{\mathscr{N}}_\ell)(\bar{k})$, and thus we can decompose the identity morphism on $H_\ell(M)$ as $T\circ\mathrm{Frob}_x$. Therefore, Frob_x^{-1} acts as T on $H_\ell(M)$, and we obtain

$$\det_{\mathbf{A}_\ell}(1 - Z^{d_x}\cdot\mathrm{Frob}_x^{-1} \mid H_\ell(M)^{I_K}) = \det_{\mathbf{A}_\ell\otimes k}(1 - Z^{d_x}\cdot\tau^{d_x} \mid \bar{\mathscr{N}}_\ell).$$

We want to show that $\overline{\mathscr{N}}_\ell = \overline{(\hat{\mathscr{M}}_\ell)_1}$. As the module $\hat{\mathscr{N}}_\ell$ is the maximal smooth sub-τ-sheaf of $\hat{\mathscr{M}}_\ell$ on $\hat{C}_{R,\ell}$, we have

$$(\hat{\mathscr{M}}_\ell)_1 \subset \hat{\mathscr{N}}_\ell.$$

By Theorem 2.1, the nondegenerate rank of $\hat{\mathscr{N}}_\ell$ is at most r', which shows that

$$\hat{\mathscr{N}}_\ell = (\hat{\mathscr{M}}_\ell)_1$$

as both are saturated smooth ℓ-adic sub-τ-sheaves on $\hat{C}_{R,\ell}$ of rank r' of $\hat{\mathscr{M}}_\ell$.

Finally, we can deduce that, for all closed points $\ell \in C''$,

$$\begin{aligned} L_x(H_\ell(M); Z)^{-1} &:= \det_{\mathbf{A}_\ell}(1 - Z^{d_x} \cdot \mathrm{Frob}_x^{-1} \mid H_\ell(M)^{I_K}) \\ &= \det_{\mathbf{A}_\ell \otimes k}(1 - Z^{d_x} \cdot \tau^{d_x} \mid \bar{\mathcal{N}}_\ell) \\ &= \det_{\mathbf{A}_\ell \otimes k}(1 - Z^{d_x} \cdot \tau^{d_x} \mid \overline{(\hat{\mathcal{M}}_\ell)_1}) \\ &= L_x(M; Z)^{-1}. \qquad \square \end{aligned} \tag{13}$$

8. Reduction of Drinfeld modules

8.1. Drinfeld modules and t-motives

Let us recall how Drinfeld modules and t-motives are related to each other (cf. [La] Appendix A). Let C be an irreducible smooth projective curve minus one closed point, and $\mathbf{A}$ its ring of global regular functions. For an $\mathbb{F}_q$-field K, we consider a Drinfeld $\mathbf{A}$-module

$$\phi : \mathbf{A} \to \mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_{a,K})$$

whose characteristic homomorphism is the homomorphism $\iota^* : \mathbf{A} \to K$ induced by ι. The K-vector space

$$M(\phi) := \mathrm{Hom}_{\mathbb{F}_q}(\mathbb{G}_{a,K}, \mathbb{G}_{a,K})$$

is endowed with the structure of an $\mathbf{A} \otimes K$-module via

$$(a \otimes c) \cdot m := cm \circ a,$$

for all $m \in M(\phi)$, $c \in K$ and $a \in \mathbf{A}$. Drinfeld proved that $M(\phi)$ is a projective $\mathbf{A} \otimes K$-module, whose rank equals the rank of ϕ as a Drinfeld module.

The endomorphism $\tau \in \mathrm{End}(\mathbb{G}_{a,K})$ acting on $M(\phi)$ via $\tau \cdot m := \tau \circ m$ commutes with $\mathbf{A}$ and acts as Frobenius on K. We can now view $M(\phi)$ as a locally free coherent sheaf over C_K, called the *t-motive associated to* ϕ, and the injective morphism $\tau : \sigma^* M(\phi) \to M(\phi)$ endows $M(\phi)$ with the structure of a τ-sheaf on C_K with characteristic ι. The category of Drinfeld modules and that of the associated t-motives are antiequivalent. Also, for every maximal ideal ℓ of $\mathbf{A}$ (or the associated closed point of C) different from the characteristic ideal $\ker \iota^*$, the Tate module $T_\ell(\phi)$ (defined as the projective limit of ℓ-primary torsion points) is isomorphic to $T_\ell(M(\phi))$ as an $\mathbf{A}_\ell[\Gamma_K]$-module.

8.2. Analytic structure of t-motives of Drinfeld modules

Let R be a complete discrete valuation $\mathbb{F}_q$-algebra with a perfect residue field k. Alongside the notion of a good model for τ-sheaves, there is the following well-known definition of *good reduction* for Drinfeld modules in case the valuation of K is finite. A model for a Drinfeld module ϕ (of rank r) is an isomorphic Drinfeld module with coefficients in R, and ϕ is said to have good reduction over K if it has a model whose reduction $\bar{\phi}$ modulo the maximal ideal of R (up to isomorphism of the choice of the model) is a Drinfeld module of the same rank r, over the residue field k of R. The following theorem explains how, fortunately enough, the two concepts of good reduction of Drinfeld modules and τ-sheaves coincide, and describes the structure of $M(\phi)$ if ϕ does not have good reduction:

Theorem 8.1. *Let ϕ be a Drinfeld* **A**-*module defined over K.*

(i) *Suppose that the valuation of K is finite with respect to ι. Then ϕ has good reduction over R if and only if its t-motive $M(\phi)$ has a good model on C_R.*

(ii) *Suppose that the valuation of K is finite with respect to ι and that ϕ does not have potential good reduction* (*We recall that a property, related to a field K, holds* potentially, *if it holds for some finite extension K' of K*). *Then there exists*

- *a finite extension K' of K* (*with valuation ring R'*),
- *a Drinfeld module ϕ' over K' of rank $\bar{r}$ with good reduction* (*by i*), *its t-motive $M(\phi')$ has a good model over $C_{R'}$*) *and*
- *a trivial τ-sheaf $\mathcal{N}$ on $C_{R'}$ of rank $r-\bar{r}$* (*we recall that a τ-sheaf is called* trivial *if it is isomorphic to a direct sum of copies of the unit τ-sheaf with underlying sheaf the structural sheaf and endomorphism τ satisfying* $\tau(1)=1$ *for the unit section* 1) *such that*,

if we denote the generic fibre of $\mathcal{N}$ by N, and the analytic τ-sheaves (cf. *Section* 6) *on $\tilde{C}_{K'}$* (*resp.* $\tilde{C}_{R'}$) *associated with $M(\phi), M(\phi')$ and N* (*resp.* $\mathcal{M}(\phi)$ *and* $\mathcal{N}$) *by $\tilde{M}(\phi)$, $\tilde{M}(\phi')$ and $\tilde{N}$* (*resp.* $\tilde{\mathcal{M}}(\phi)$ *and* $\tilde{\mathcal{N}}$),
then there exists an exact sequence

$$0 \to \tilde{N} \to \tilde{M}(\phi) \to \tilde{M}(\phi') \to 0 \tag{14}$$

of τ-sheaves on $\tilde{C}_{K'}$, and $\tilde{\mathcal{N}}$ is the maximal good sub-τ-sheaf of $\tilde{\mathcal{M}}(\phi)$.

(iii) *Suppose that the valuation of K is infinite with respect to ι. Then there exists a finite totally ramified extension K' of K* (*with valuation ring R'*) *such that $M(\phi)$ has a good model over $C_{R'}$ whose reduction is a trivial τ-sheaf on C_k.*

Proof. (i) Takahashi's Galois criterium for good reduction (cf. [Ta]) says that a Drinfeld module ϕ has a model with good reduction if and only if there exists a maximal ideal $\ell \neq \ell_x$ of $\mathbf{A}$ such that the Tate module $T_\ell(\phi)$ is unramified and, by our Theorem 1.1, the latter holds if and only if the t-motive $M(\phi)$ has a good model over C_R.

(ii) If ϕ does not have potential reduction, then it has potential Tate reduction. The statement is then a consequence of Drinfeld's theorem on Tate uniformization for Drinfeld modules, as we proved in [Ga3] Theorem 1.2 + Example 7.1.1.

(iii) All Drinfeld modules being uniformizable at infinity, this follows from [Ga2] Main Theorem. □

8.3. Reduction of t-motives of Drinfeld modules

For a Drinfeld $\mathbf{A}$-module ϕ over K of rank r and its t-motive $M(\phi)$ on C_K, let $\mathscr{M}$ denote the maximal model over C_R of $M(\phi)$, $\bar{\mathscr{M}}$ its reduction to C_k. Let $\bar{\mathscr{M}}_1$ be the maximal τ-sheaf contained in $\bar{\mathscr{M}}$ (cf. (12)).

Theorem 8.2.

(i) *Suppose that the valuation of K with respect to $\imath$ is finite and that the Drinfeld module ϕ has good reduction over K. Then $\bar{\mathscr{M}} = \bar{\mathscr{M}}_1$ is isomorphic to the t-motive associated to the Drinfeld module $\bar{\phi}$ over k.*

(ii) *Suppose that the valuation of K with respect to $\imath$ is finite and that ϕ has no good reduction over K, but has potential good reduction. Then $\bar{\mathscr{M}}_1 = 0$.*

(iii) *If either the valuation of K with respect to $\imath$ is finite and the Drinfeld module ϕ has no potential good reduction, or the valuation of K with respect to $\imath$ is infinite, then $\bar{\mathscr{M}}_1 = 0$ or $\bar{\mathscr{M}}_1$ is a potentially trivial τ-sheaf on C_k.*

First, we mention the following easy lemma:

Lemma 8.3. *Suppose K' is a finite extension of K, with residue field k'. If we define $\mathscr{M}'$, $\bar{\mathscr{M}}'$ and $\bar{\mathscr{M}}'_1$ as above, then $\bar{\mathscr{M}}_1 \otimes_k k' \hookrightarrow \bar{\mathscr{M}}'_1$.*

Proof of Theorem 8.2. (i) We can easily reduce ourselves to the case $C = \mathbb{A}^1$, i.e. $\mathbf{A} = \mathbb{F}_q[t]$. Let $x \in M(\phi)$ denote the identity morphism $id : \mathbb{G}_{a,K} \to \mathbb{G}_{a,K}$. The elements $m_i := \tau^{i-1} \circ x$ for $i = 1, \dots, r$ yield a basis for the $\mathbf{A} \otimes K$-module $M(\phi)$ (or, equivalently, for the coherent free sheaf $M(\phi)$ on $\mathbb{A}^1_K$). If the Drinfeld module ϕ is given by

$$\phi : t \mapsto \sum_{i=0}^{r} a_i \tau^i \in \mathrm{End}(\mathbb{G}_{a,K}),$$

(where $a_r \in K^\times$ and $a_0 = \imath^*(t)$), then the action of τ with respect to this basis is given by the matrix representation:

$$\tau \cdot (m_1, \ldots, m_r) = (m_1, \ldots, m_r) \cdot \begin{pmatrix} 0 & \ldots & 0 & \dfrac{t-a_0}{a_r} \\ 1 & & 0 & -\dfrac{a_1}{a_r} \\ & \ddots & & \vdots \\ 0 & & 1 & -\dfrac{a_{r-1}}{a_r} \end{pmatrix}. \tag{15}$$

If ϕ has good reduction, then, upon replacing ϕ by a model with good reduction, we may assume that $a_i \in R$, for all $i = 1, \ldots, r$ and that $a_r \in R^\times$. This shows that the $\mathcal{O}_{C_R}$-sheaf $\mathcal{M}$ generated by the elements m_i is a good model for the τ-sheaf $M(\phi)$ (hence isomorphic to the maximal model $\mathcal{M}$). Reducing Eq. (15) yields that $\bar{\mathcal{M}}$ is isomorphic to $M(\bar{\phi})$.

(ii) Suppose that ϕ has good reduction over a finite extension K' of K. By the lemma, $\bar{\mathcal{M}}_1 \otimes_k k'$ injects into $\bar{\mathcal{M}}'$. On the other hand, the τ-sheaf $\bar{\mathcal{M}}'$ is the t-motive of a Drinfeld module $\bar{\phi}$, by (i), and is therefore a simple τ-sheaf, i.e. contains no nontrivial sub-τ-sheaves of smaller rank (this follows e.g. from the purity of the Drinfeld module (cf. [An1] Proposition 4.1.1). Hence, if $\bar{\mathcal{M}}_1 \neq 0$, then $\mathcal{M}$ is a good model for $M(\phi)$, and therefore ϕ must have good reduction over K; this concludes the proof of (ii).

(iii) Suppose that the valuation of K with respect to $\imath$ is finite and the Drinfeld module ϕ has no potential good reduction, or that the valuation of K with respect to $\imath$ is infinite. Let K' be an extension of K as in Theorem 8.1 (ii), resp. (iii). By Lemma 8.3, $\bar{\mathcal{M}}_1 \otimes_k k'$ injects into $\bar{\mathcal{M}}'$, which, by that theorem, is a trivial τ-sheaf on C_k. This implies, e.g. by the theory of weights of Dieudonné modules and weights explained in [La] Appendix B, that $\bar{\mathcal{M}}_1$ is either zero or potentially trivial. $\square$

8.4. *L-factors for Drinfeld modules*

We assume that the residue field k of K is finite. Let ϕ be a Drinfeld $\mathbf{A}$-module with good reduction. The *characteristic polynomial of Frobenius at x*, defined as

$$P_{\phi,x} := \det_{\mathbf{A}_\ell}(1 - Z^{d_x} \cdot \mathrm{Frob}_x | T_\ell(\phi)) \in \mathbf{A}_\ell[Z]$$

for $\ell \neq \ell_x$, has coefficients in $\mathbf{A}$ and is independent of ℓ. The purity of its roots and many other interesting results can be found in [Go2] Theorem 3.2.3.

Using Theorem 8.2, we are now able to give a detailed description of the local L-factor $L_x(M(\phi); Z)^{-1}$ for every possible type of reduction:

Corollary 8.4.

(i) *Suppose that the valuation of K with respect to ι is finite and that the Drinfeld module ϕ has good reduction over K. Then*
$$L_x(M(\phi); Z)^{-1} = P_{\phi,x}(Z).$$

(ii) *Suppose that the valuation of K with respect to ι is finite and that ϕ has no good reduction over K, but has potential good reduction. Then*
$$L_x(M(\phi); Z)^{-1} = 1.$$

(iii) *If either the valuation of K with respect to ι is finite and the Drinfeld module ϕ has no potential good reduction, or the valuation of K with respect to ι is infinite, then the local L-factor of ϕ has constant coefficients*:
$$L_x(M(\phi); Z)^{-1} \in \mathbb{F}_q[Z].$$

Proof. Statements (ii) and (iii) are immediately clear from Theorem 8.2. For (i), we notice that, by Theorem 8.2(i), $T_\ell(\bar{\phi}) = T_\ell(\bar{\mathscr{M}})$, and that, as $T_\ell(\bar{\mathscr{M}}) = H_\ell(M)^*$,
$$L_x(M(\phi); Z)^{-1} = \det_{\mathbf{A}_\ell}(1 - Z^{d_x} \cdot \mathrm{Frob}_x | T_\ell(\bar{\mathscr{M}})),$$
and this for all $\ell \neq \ell_x$. $\square$

Remark 8.5. This result looks analogous to that in the case of elliptic curves: Let E be an elliptic curve defined over a number field $K/\mathbb{Q}$, and let x be a nonarchimeadian place of K. At places of bad reduction, the poles of the local L-factor $L_x(E, Z)$ are roots of unity:

- If E has additive reduction, then $L_x(E, Z)^{-1} = 1$.
- If E has multiplicative reduction, then $L_x(E, Z)^{-1} = 1 \pm Z$.

9. Models for T-modules

In [An1], Anderson developed a theory of higher dimensional t-modules. We want to indicate here why this generalization seems to be less successful in this context of reduction of t-modules than it has been elsewhere. Taking $C := \mathbb{A}^1$ and putting $E := \mathbb{G}_{a,K}^{\otimes d}$, a *t-module* (E, ϕ_E) (for short: E) is a ring homomorphism
$$\phi_E : \mathbf{A} \to \mathrm{End}_{\mathbb{F}_q}(E).$$

Such a t-module is called *abelian*, with characteristic homomorphism $\imath^*$: $\mathbf{A} \to K$, if

- the endomorphism $\phi_E(t)$ operates with the single eigenvalue $\imath^*(t)$ on the tangent space $\mathrm{Lie}(E)$, and
- the K-vector space

  $$M(E) := \mathrm{Hom}_{\mathbb{F}_q}(E, \mathbb{G}_{a,K})$$

  is a finitely generated $\mathbf{A} \otimes K$-module, the action of $\mathbf{A}$ being induced by that on E.

For such an abelian t-module E, the $\mathbf{A} \otimes K$-module $M(E)$ is actually free (its rank is called the *rank of* E), and it is endowed with an injective semilinear endomorphism given by Frobenius on $\mathbb{G}_{a,K}$. Thus, $M(E)$ can be viewed as a τ-sheaf on $\mathbb{A}^1_K$, called the *t-motive of E*. For every maximal ideal ℓ of $\mathbf{A}$ (or the associated closed point of C) different from the characteristic ideal $\ker \imath^*$, the Tate module $T_\ell(E)$ (defined as the projective limit of ℓ-primary torsion points) is isomorphic to $T_\ell(M(E))$ as an $\mathbf{A}_\ell[\Gamma_K]$-module.

Suppose that the valuation of K is finite with respect to $\imath^*$. The following notion of good reduction seems natural: a model for an abelian t-module E (of rank r) is an isomorphic t-module E' such that $\phi_{E'}$ has coefficients in R, and E is said to have *good reduction over* K if it has a model whose reduction $\bar{E}'$ modulo the maximal ideal of R is an abelian t-module of the same rank r, over the residue field k of R. If E has good reduction over K, then $T_\ell(E)$ is unramified; hence we obtain the following result as a corollary of Theorem 1.1: if E has good reduction over K, then its t-motive $M(E)$ has a good model on $\mathbb{A}^1_K$.

Contrary to the case of Drinfeld modules, however, it is not at all clear if and how the converse statement could hold. In fact, it is troublesome to try to develop a reduction theory in the category of abelian t-modules. Let us look at the notion of good reduction for an abelian t-module E: except if E is a Drinfeld modules (cf. above), we cannot test if a model for E has good reduction just by looking at the reduced t-module $\bar{E}'$ itself, but we need to check whether the reduction is really abelian and whether it has the appropriate rank.

There is also little hope for a fruitful theory of reduction for higher dimensional abelian t-modules without good reduction. An essential step would be to find a model $\mathscr{E}$ over R whose reduction $\bar{\mathscr{E}}$ is nondegenerate in the following sense: there exists a sub-t-module $\bar{\mathscr{E}}_1$ of $\bar{\mathscr{E}}$ which is abelian. As a consequence, the Tate module $T_\ell(E)$ would contain a non-trivial unramified submodule for all closed points $\ell \neq \ell_x$ of $\mathbb{A}^1$, corresponding to the Tate module

$$T_\ell(\bar{\mathscr{E}}_1) \hookrightarrow T_\ell(E).$$

Let R denote the discrete valuation ring $R := \mathbb{F}_q[[\pi]]$, and K its field of fractions. Consider, for some $\gamma \in K$ with $v(\gamma) < 0$, the τ-sheaf $M(\gamma)$ on $\mathbb{A}^1_K$ with underlying sheaf $\mathcal{O}_{\mathbb{A}^1_K} \oplus \mathcal{O}_{\mathbb{A}^1_K}$, with global basis

$$\mathbf{m} = (m_1, m_2),$$

and set τ to be given by the matrix representation

$$\tau(\mathbf{m}) = \mathbf{m} \cdot \begin{pmatrix} 0 & t-\theta \\ t-\theta & \gamma t \end{pmatrix}. \tag{16}$$

In [Ga1] Lemma 5.12, we proved that for $M(\gamma)$ the Tate module $T_{\ell_0}(M(\gamma))$ (with $\ell_0 \neq \ell_x$) does not potentially contain an unramified submodule.

Actually, $M(\gamma)$ is a pure t-motive of rank 2, dimension 2 and weight 1. If we take (m_1, m_2) to be the coordinate functions on $\mathbb{G}_{a,K}^{\oplus 2}$, then the t-module $E(\gamma)$ is given by

$$(t_E - \theta)\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} -\gamma\tau & \tau - \theta\gamma \\ \tau & 0 \end{pmatrix}\begin{pmatrix} m_1 \\ m_2 \end{pmatrix}.$$

Thus, we see that there cannot, not even potentially, exist a model $\mathscr{E}$ for $E(\gamma)$ which is nondegenerate.

In conclusion, this example rules out the possibility of a decent reduction theory for abelian t-modules. The arithmetic study of general abelian t-modules should therefore rely on that of the associated τ-sheaf $M(E)$!

An instance of this idea is the following: assume that (the analytic τ-sheaf associated to) a pure t-motive $M(E)$ is strongly semistable and that all the subquotients of a semistable filtration for $M(E)$ are pure. By an equivalence between analytic morphism of pure t-motives and pure t-modules which we prove in [Ga3] Theorem 1.3, this then yields an analytic description of E analogous to the Tate uniformization theorem for Drinfeld modules (cf. [Dr] Section 5).

Acknowledgments

I thank my promotors Richard Pink and Jan Van Geel for their many helpful interventions and constant support. I am also indebted to Gebhard Böckle and David Goss for a lot of interesting remarks and suggestions concerning the local L-factors.

References

[An1] G.W. Anderson, t-Motives, Duke Math. J. 53 (1986) 457–502.
[An2] G.W. Anderson, On Tate modules of formal t-modules, Intern. Math. Res. Notices 2 (1993) 41–53.

[Bo] G. Böckle, Global L-functions over function fields, Math. Ann. 4 (2002) 737–795.
[BL] S. Bosch, W. Lütkebohmert, Formal and rigid geometry I, Math. Ann. 295 (1993) 291–317.
[BLR] S. Bosch, W. Lütkebohmert, M. Raynaud, Néron Models, Ergebnisse der Mathematik 21, Springer-Verlag, Berlin, 1990.
[Dr] V.G. Drinfeld, Elliptic modules, Math. U.S.S.R. Sbornik 23 (1976) 561–592.
[Ga1] F. Gardeyn, The analytic structure of τ-sheaves, preprint (2001), cf. http://www.math.ethz.ch/˜fgardeyn, J. Number Theory, in press.
[Ga2] F. Gardeyn, New criteria for Anderson uniformizability of t-motives, preprint (2001), cf. loc. cit.
[Ga3] F. Gardeyn, Analytic morphisms of t-motives, preprint (2001), cf. loc. cit.
[Go1] D. Goss, Basic Structures of Function Field Arithmetic, Ergebnisse der Mathematik 35, Springer, Berlin, 1996.
[Go2] D. Goss, L-series of t-motives and Drinfeld modules, in: The Arithmetic of Function Fields, Proceedings of the workshop at Ohio State University 1991, Walter de Gruyter, Berlin, New York, 1992, pp. 313–402.
[La] G. Laumon, Cohomology of Drinfeld modular varieties I, in: Cambridge Studies in Advanced Mathematics, Vol. 41, Cambridge University Press, Cambridge, 1996.
[Ta] T. Takahashi, Good reduction of elliptic modules, J. Math. Soc. Jpn. 34/3 (1982) 475–487.
[TW] Y. Taguchi, D. Wan, L-functions of φ-sheaves and Drinfeld modules, J. AMS 9/3 (1996) 755–781.